

Non-Split Geometry on Products of Vector Bundles

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Abstract

We propose a model in which a spliced vector bundle (with an arbitrary number of gauge structures in the splice) possesses a geometry which do not split. The model employs connection 1-forms with values in a space-product of Lie algebras, and therefore interlaces the various gauge structures in a non-trivial manner. Special attention is given to the structure of the geometric ghost sector and the super-algebra it possesses: The ghosts emerge as x -dependent deformations at the gauge sector, and the associated BRST super algebra is realized as constraints that follow from the invariance of the curvature.

1 Introduction

A product of vector bundles, within the classical framework of gauge theories, is often contemplated as the bundle of product-space fibers, where each factor-fiber in the splice is a representation space for a certain gauge group. This results in a geometrical splitting by the following means: When the geometrical aspects of a single group structure are considered, those components of a geometric object that correspond to other coexisting group structures, all remain non-active. This fact is after all a consequence of the Leibniz rule. For example: The

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absolute differential of a tensor product of two fiber bases splits into a sum of tensor products of single-basis differentials, which are in turn used to define the corresponding factor structure connections,

$$\begin{aligned} d(\mathbf{e}_1 \otimes \mathbf{e}_2) &= (d\mathbf{e}_1) \otimes \mathbf{e}_2 + \mathbf{e}_1 \otimes d(\mathbf{e}_2) \\ &=: -\omega_1(\mathbf{e}_1) \otimes \mathbf{e}_2 - \mathbf{e}_1 \otimes \omega_2(\mathbf{e}_2). \end{aligned} \tag{1}$$

This splitting, however, is not compatible with the concepts of fusion and unification. The following question therefore arises: Is it possible to form a better glue of symmetry structures, a one that results in a single non-split geometry of the composite bundle? In this article we provide an affirmative answer to this question. We shall replace definition (1) with a somewhat less intuitive definition,

$$\begin{aligned} d(\mathbf{e}_1 \otimes \mathbf{e}_2) &= (d\mathbf{e}_1) \otimes \mathbf{e}_2 + \mathbf{e}_1 \otimes d(\mathbf{e}_2) \\ &=: -\omega_1(\mathbf{e}_1 \otimes \mathbf{e}_2) - \omega_2(\mathbf{e}_1 \otimes \mathbf{e}_2), \end{aligned} \tag{2}$$

and discuss the conditions under which it is really meaningful. This will lead to a geometry which do not split even though the bundle itself inherently splits.

Our model is based on a collection of connection 1-forms, each taking values in a space product of Lie algebras, instead of in a single Lie algebra. These are later integrated to form a single curvature of the multi-structure splice, and the latter obeys the Bianchi identity with respect to an appropriately-constructed covariant exterior derivative.

We shall also derive the associated ghost structure and BRST symmetry by pure geometrical means: The connections undergo a (local) deformation, and the basespace is extended by multiplying it (locally again) with the spaces spanned by the symmetry groups. The exterior derivatives in group-spaces are then identified with the BRST coboundary operators, and the deformation elements at the gauge sector are identified with the ghosts. The BRST algebra then emerges as structural constraints that follow directly from the demand that the curvature of the whole splice will remain intact.

We refer the reader to the following physically-inclined accounts as a background material for this article: A concise presentation of the concept of fiber bundles is given in [1] p. 95-117; the concept of a product bundle, where distinct symmetry structures share a common basespace, is discussed in [2] p. 194-196. A detailed presentation and analysis of ghosts and BRST symmetries from the field theory point of view is found in [3] p. 141-181. The same subject, presented from the geometrical perspective (more relevant to our proposes) is given in [1], chapters 8 & 9.

Notations and Conventions:

We shall consider a product bundle that hosts an arbitrary (but finite) number of coexisting symmetry structures, say m . The underlying manifold M is smooth and oriented. \mathcal{B}_x is the basespace at x , \mathcal{F}_x is the local fiber. It consists of a product space of m vector spaces $\{V_\alpha\}$, each of which is acted upon by a specific gauge group $G_\alpha(x)$. Here and in the following, $\alpha, \gamma, \dots = 1, \dots, m$ label the members of the hosted vector spaces, the associated symmetry groups, and their generating Lie algebras; $n = \dim M$, $n_\alpha = \dim G_\alpha$, $N_\alpha = \dim V_\alpha$.

Our conventions with respect to indices goes as follows: $a_\alpha, b_\alpha, \dots = 1 \cdots n_\alpha$ are G_α -indices (associated with the symmetry group G_α). $A_\alpha, B_\alpha, \dots = 1 \cdots N_\alpha$ are V_α -fiberspace indices (associated with the representation space V_α of G_α). The basespace employs Greek indices, $\mu, \nu, \dots = 1 \cdots n$ which are, without lose of generality, taken to be holonomic. Concerning with brackets notation, for any p -form ψ , and q -form ϕ ,

$$[\psi, \phi]_{\mp} := \psi \wedge \phi \mp \phi \wedge \psi, \quad (3)$$

$$[\![\psi, \phi]\!] := \psi \wedge \phi - (-1)^{pq} \phi \wedge \psi. \quad (4)$$

2 The foliar bundle and its associated curvature

Let F refer to a product bundle which consists of m distinct independent coexisting gauge structures. The elements $\{L^\alpha\}$ of the algebra $\text{Lie } G_\alpha$ that generates G_α (for any $\alpha = 1 \cdots m$),

are assumed to carry a faithful representation ρ_α in V_α , and to extend to the full enveloping algebra, so their anti-commutator is well defined. In what follows we shall restrict ourselves to deal only with G_α -structures that possess representations in which

$$[\rho_\alpha(L^\alpha), \rho_\alpha(L^\alpha)]_+ \in \text{span} \{ \rho_\alpha(L^\alpha) \}; \quad (5)$$

namely, *the realizations of the algebras close with respect to anti-commutation*. An algebra whose elements in a certain representation close with respect to anti-commutation is said to be *sealed* in that representation. The requirement that the algebra be sealed in a representation is obligatory to our present purposes. A simple example of such an algebra is the one which generates invertible linear transformations in a vector space, $\mathfrak{gl}(n, \mathbb{R})$. Concerning with unitary gauge structures, the closure relation (5), can be realized only if the algebra is extended by central elements.

Let us now introduce a set of m G_α -induced connection 1-forms $\{\omega_\alpha\}$ with values in the *symmetric* product-space $\bigotimes_{\alpha=1}^m (\text{Lie } G_\alpha)$,

$$\omega_\alpha =: \rho_F(\tilde{\omega}_\alpha) =: \tilde{\omega}_\mu^{a_1 \dots a_m}(G_\alpha) dx^\mu \rho_{a_1 \dots a_m}, \quad (6)$$

where $\{\tilde{\omega}^{a_1 \dots a_m}(G_\alpha)\}$ are those connection coefficients which are associated with the gauge group G_α , the short-hand writing $\rho_{a_1 \dots a_m}$ stands for the (symmetric) product of matrices,

$$(\rho_{a_1 \dots a_m})_{A_1 \dots A_m}^{B_1 \dots B_m} := \bigotimes_{\alpha=1}^m (\rho_\alpha(L_{a_\alpha}^\alpha))_{A_\alpha}^{B_\alpha}, \quad (7)$$

and we defined $\rho_F(\varpi) := \varpi^{a_1 \dots a_m} \rho_{a_1 \dots a_m}$ for any $\varpi \in \bigotimes_{\alpha=1}^m (\text{Lie } G_\alpha)$. Notice that, in general, $\tilde{\omega}^{a_1 \dots a_m}(G_\alpha) \neq \tilde{\omega}^{a_1 \dots a_m}(G_\gamma)$ for $\alpha \neq \gamma$, hence $\omega_\alpha \neq \omega_\gamma$.

Under a gauge transformation (see also the discussion concerning with eq. (19)), each element in the collection $\{\omega_\alpha\}$ should transform as:

$$\forall g_\gamma \in G_\gamma : \begin{cases} \omega_\alpha \mapsto g_\gamma(\omega_\alpha + d)g_\gamma^{-1} & \gamma = \alpha \\ \omega_\alpha \mapsto g_\gamma \omega_\alpha g_\gamma^{-1} & \gamma \neq \alpha \end{cases} \quad (8)$$

where the actions of the g 's are given by means of matrix multiplication. Each ω_α , therefore, transforms as a connection with respect to its inducing gauge group G_α , but it behaves as a tensor with respect to the rest of the groups in the collection.

The set of connection 1-forms introduced above is seen to give rise to a unique curvature 2-form which characterizes the whole splice:

$$\mathcal{R}_F(\omega_1, \dots, \omega_m) = \sum_{\alpha, \gamma=1}^m (d\omega_\alpha + \omega_\alpha \wedge \omega_\gamma). \quad (9)$$

To see that this is indeed a “proper” curvature, we follow two steps:

First we verify that \mathcal{R}_F as well takes values $\in \bigotimes_\alpha (\text{Lie } G_\alpha)$. This however follows directly from our previous demand, eq. (5), that $\rho_\alpha(\text{Lie } G_\alpha)$ (for any $\alpha = 1 \dots m$) closes with respect to anti-commutation. In this case,

$$\rho_\alpha(L^\alpha) \rho_\alpha(L^\alpha) = \frac{1}{2} [\rho_\alpha(L^\alpha), \rho_\alpha(L^\alpha)]_+ + \frac{1}{2} [\rho_\alpha(L^\alpha), \rho_\alpha(L^\alpha)]_- \subset \text{span} \{ \rho_\alpha(\text{Lie } G_\alpha) \}. \quad (10)$$

Since each term in $\sum_{\alpha, \gamma} \omega_\alpha \wedge \omega_\gamma$ contains products of pairs of generators (the elements in each pair belong to the same Lie algebra), assignment (10) guarantees that the resulting algebraic expansion will always lay in $\bigotimes_\alpha (\text{Lie } G_\alpha)$. Hence,

$$\begin{aligned} \mathcal{R}_F &:= \mathcal{R}_{\mu\nu}(\rho_F(\tilde{\omega})) dx^\mu \wedge dx^\nu = \sum_{\alpha, \gamma=1}^m [d\rho_F(\tilde{\omega}_\alpha) + \rho_F(\tilde{\omega}_\alpha) \wedge \rho_F(\tilde{\omega}_\gamma)] \\ &= \sum_{\alpha, \gamma=1}^m d\rho_F(\tilde{\omega}_\alpha) + \rho_F(\tilde{\omega}_\alpha \wedge \tilde{\omega}_\gamma) = \rho_F(\mathcal{R}_{\mu\nu}(\tilde{\omega})) dx^\mu \wedge dx^\nu, \end{aligned} \quad (11)$$

where $\rho_F(\mathcal{R}) = \mathcal{R}^{a_1 \dots a_m} \rho_{a_1 \dots a_m}$. We shall adopt the shortage notation: $a_1 \dots a_m = \{a\}$ etc.

Then, $\rho_F(\mathcal{R}) = \mathcal{R}^{\{a\}} \rho_{\{a\}}$, and

$$\mathcal{R}^{\{a\}} = \sum_{\alpha=1}^m d\tilde{\omega}_\alpha^{\{a\}} + \sum_{\alpha, \gamma=1}^m f^{\{a\}\{b\}\{c\}} \tilde{\omega}_\alpha^{\{b\}} \wedge \tilde{\omega}_\gamma^{\{c\}}, \quad (12)$$

where $f^{\{a\}\{b\}\{c\}}$ is defined through $[\rho_{\{a\}}, \rho_{\{b\}}]_- = f^{\{a\}\{b\}\{c\}} \rho_{\{c\}}$.¹

¹ Formula (12) is coming from:

$$\sum_{\alpha, \gamma} \omega_\alpha \wedge \omega_\gamma = \frac{1}{2} \sum_{\alpha, \gamma} \sum_{\{a\}}^{\{n\}} \sum_{\{b\}}^{\{n\}} \left[(\tilde{\omega}_\alpha)_\mu^{\{a\}} \rho_{\{a\}}(\tilde{\omega}_\gamma)_\nu^{\{b\}} \rho_{\{b\}} - (\tilde{\omega}_\alpha)_\nu^{\{a\}} \rho_{\{a\}}(\tilde{\omega}_\gamma)_\mu^{\{b\}} \rho_{\{b\}} \right] dx^\mu \wedge dx^\nu$$

Second, we should verify that \mathcal{R}_F transforms linearly (namely, as a tensor) with respect to each gauge group in the collection. Indeed, considering a particular label, say α , \mathcal{R}_F can be decomposed as,

$$\begin{aligned} & d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha \\ & + \sum_{\gamma \neq \alpha} (d\omega_\gamma + \omega_\alpha \wedge \omega_\gamma + \omega_\gamma \wedge \omega_\alpha) \\ & + \sum_{\gamma, \epsilon \neq \alpha} \omega_\gamma \wedge \omega_\epsilon \end{aligned} \quad (14)$$

which we may also write as

$$\mathcal{R}_F = \mathcal{R}_\alpha(\omega_\alpha) + \sum_{\gamma \neq \alpha} \mathcal{D}_{\omega_\alpha} \omega_\gamma + \sum_{\gamma, \epsilon \neq \alpha} \omega_\gamma \wedge \omega_\epsilon, \quad (15)$$

where $\mathcal{D}_{\omega_\alpha} \omega_\gamma$ is the covariant exterior derivative of ω_γ with respect to the connection ω_α . Under the action of G_α , each summand in (15) transforms linearly, and in a manner which is independent of all the other summands,

$$\forall g_\alpha \in G_\alpha \quad \left\{ \begin{array}{ll} R_\alpha(\omega_\alpha) & \mapsto g_\alpha R_\alpha(\omega_\alpha) g_\alpha^{-1}, \\ D_{\omega_\alpha} \omega_\gamma & \mapsto g_\alpha (D_{\omega_\alpha} \omega_\gamma) g_\alpha^{-1} \quad (\gamma \neq \alpha), \\ \omega_\gamma \wedge \omega_\epsilon & \mapsto g_\alpha (\omega_\gamma \wedge \omega_\epsilon) g_\alpha^{-1} \quad (\gamma, \epsilon \neq \alpha). \end{array} \right. \quad (16)$$

This, however, holds for any $\alpha = 1, \dots, m$; thus \mathcal{R}_F is linear with respect to all the G 's and our claim that \mathcal{R}_F is a proper curvature has been established.

In fact, $\sum_\alpha \omega_\alpha := \omega_F$ can be regarded as a single connection, having the property of simultaneously supporting many gauges:

$$\forall \alpha \ \& \ \forall g_\alpha \in G_\alpha : \quad \omega_F \mapsto g_\alpha (\omega_F + d) g_\alpha^{-1}. \quad (17)$$

$$\begin{aligned} & = \frac{1}{2} \sum_{\alpha, \gamma} \sum_{\{a\}}^{\{n\}} \sum_{\{b\}}^{\{n\}} (\tilde{\omega}_\alpha)_\mu^{\{a\}} (\tilde{\omega}_\gamma)_\nu^{\{b\}} [\rho_{\{a\}}, \rho_{\{b\}}]_- dx^\mu \wedge dx^\nu \\ & = \sum_{\alpha, \gamma} \sum_{\{a\}}^{\{n\}} \sum_{\{b\}}^{\{n\}} \sum_{\{c\}}^{\{n\}} \tilde{\omega}_\alpha^{\{a\}} \wedge \tilde{\omega}_\gamma^{\{b\}} f_{\{a\}\{b\}\{c\}} \rho_{\{c\}}. \end{aligned} \quad (13)$$

Therefore, ω_F underlies a generic formation of gauge, in which m distinct coexisting structures are intertwined, and whose associated curvature acquires a ‘single-structure’ form,

$$\mathcal{R}_F = d\omega_F + \omega_F \wedge \omega_F. \quad (18)$$

We therefore name our geometrical construction “*foliar bundle*”; it is a product-bundle whose single-foil slices are interlaced in such a way that the overall geometry do not split.

The set of connection 1-forms introduced in (6) can naturally be derived by considering the absolute differential of the multi-foil basis, $\mathbf{e}_{A_1 \dots A_m} = \bigotimes_{\alpha=1}^m \mathbf{e}_{A_\alpha}^\alpha$:

$$\begin{aligned} d\mathbf{e}_{A_1 \dots A_m} &= d\mathbf{e}_{A_1}^1 \otimes \dots \otimes \mathbf{e}_{A_m}^m + \dots + \mathbf{e}_{A_1}^1 \otimes \dots \otimes d\mathbf{e}_{A_m}^m \\ &:= \sum_{\alpha=1}^m -\rho_F(\tilde{\omega}_\alpha)^{B_1 \dots B_m}_{A_1 \dots A_m} \mathbf{e}_{B_1 \dots B_m}, \end{aligned} \quad (19)$$

which is conveniently abbreviated as $d\mathbf{e}_{A_1 \dots A_m} = -\sum_\alpha (\omega_\alpha \mathbf{e})_{A_1 \dots A_m}$. Definition (19) remains valid in any gauge (in any symmetry slice) provided that the ω_α ’s transforms as in eq. (8).

Additional application of d on the multi-foil basis gives

$$\begin{aligned} dd\mathbf{e}_{A_1 \dots A_m} &= \left[-\sum_\alpha d\rho_F(\tilde{\omega}_\alpha)^{B_1 \dots B_m}_{A_1 \dots A_m} - \sum_\alpha \sum_\gamma \rho_F(\tilde{\omega}_\alpha)^{C_1 \dots C_m}_{A_1 \dots A_m} \wedge \rho_F(\tilde{\omega}_\gamma)^{B_1 \dots B_m}_{C_1 \dots C_m} \right] \mathbf{e}_{B_1 \dots B_m} \\ &= -(\mathcal{R}_F \mathbf{e})_{A_1 \dots A_m}. \end{aligned} \quad (20)$$

Consider the $(\times_\gamma G_\gamma)$ -tensor object Ψ_T (it transforms as a tensor with respect to any of the G ’s). By formula (8), its (foliar) covariant exterior derivative,

$$\mathcal{D}\Psi_T := d\Psi_T + \sum_\alpha \left(\omega_\alpha \wedge \Psi_T + (-1)^{\deg(\Psi_T)+1} \Psi_T \wedge \omega_\alpha \right) \quad (21)$$

$$= d\Psi_T + \omega_F \wedge \Psi_T + (-1)^{\deg(\Psi_T)+1} \Psi_T \wedge \omega_F = d\Psi_T + \llbracket \omega_F, \Psi_T \rrbracket \quad (22)$$

is a $(\times_\gamma G_\gamma)$ -tensor as well; namely $\mathcal{D}\Psi_T$ transforms as a tensor with respect to any of the G ’s. In terms of this covariant exterior derivative, the foliar curvature can be re-constructed via $\mathcal{D}\mathcal{D}\Psi_T = [\mathcal{R}_F, \Psi_T]_-$. Moreover, by the (graded) Jacobi identity,

$$\begin{aligned} 0 &= \llbracket \mathcal{D}, \llbracket \mathcal{D}, \mathcal{D} \rrbracket \rrbracket \Psi_T = 2\mathcal{D}[\mathcal{R}_F, \Psi_T]_- - 2[\mathcal{R}_F, \mathcal{D}\Psi_T]_- \\ &= 2 \left[(\mathcal{D}\mathcal{R}_F) \wedge \Psi_T - (-1)^{\deg \Psi_T} \Psi_T \wedge \mathcal{D}\mathcal{R}_F \right], \end{aligned} \quad (23)$$

and the foliar counterpart of Bianchi's identity,

$$\mathcal{DR}_F = 0, \quad (24)$$

follows immediately. Of course, this result follows directly also from eqs. (18) and (22).

3 The BRST super structure of the foliar bundle

Our present aim is to explore the geometry induced along the vertical directions (those that are parallel to the fibers). Consider the following set of m mutually-independent *horizontal* deformations

$$\omega_\alpha \rightarrow \omega_\alpha + \Omega_\alpha, \quad \alpha = 1, \dots, m, \quad (25)$$

where the deformation elements $\{\Omega_\alpha\}$ are linear with respect to all the G 's. Consequently, $\omega_\alpha + \Omega_\alpha$ transforms according to (8), but we require that ω_α and $\omega_\alpha + \Omega_\alpha$ cannot be connected through a gauge transformation; in other words, the deformations display bijections between gauge-inequivalent orbits. In general, the shifted connections correspond to a different foliar curvature. But this may be avoided according to the following prescription: One extends the basespace sector of the bundle such that it includes also the angles associated with the gauge groups, and treats these angles as if they were additional independent variables. One may then demand that the original curvature remains inert, but then he must pay a price in the form of additional structural constraints associated with the vertical directions.

Each set of angles $\{\phi^{a_\alpha}(x)\}$, which coordinating $G_\alpha(x)$, is naturally supplied with a coboundary-type operator δ_α , in complete analogy with the exterior derivative d on M . In other words, each group $G_\alpha(x)$, at any $x \in M$, associates a Grassmann space graded by δ_α . The differentiation with respect to an angle satisfies:

$$\frac{\delta \phi^{b_\alpha}}{\delta \phi^{a_\alpha}} = \delta^{a_\alpha b_\alpha}, \quad \text{or more generally,} \quad \frac{\delta \phi^{b_\gamma}}{\delta \phi^{a_\alpha}} = \delta_{\alpha\gamma} \delta^{a_\alpha b_\gamma} \quad (26)$$

because angles associated with different groups are independent of each other whatsoever. The

coboundary operator δ_α is explicitly defined through

$$\delta_\alpha \phi^{a_\alpha} := \delta \phi^{a_\alpha} \Rightarrow \delta_\alpha \equiv \delta \phi^{a_\alpha} \frac{\delta}{\delta \phi^{a_\alpha}}. \quad (27)$$

Over the extended space of differential forms,

$$\Upsilon = \Lambda^n \bigwedge_{\alpha=1}^m \Lambda^{n_\alpha}, \quad (28)$$

$\delta \phi^{a_\alpha} \wedge dx^\mu = -dx^\mu \wedge \delta \phi^{a_\alpha}$, $\delta \phi^{a_\alpha} \wedge \delta \phi^{a_\gamma} = -\delta \phi^{a_\gamma} \wedge \delta \phi^{a_\alpha}$, $\alpha, \gamma = 1, \dots, m$, $a_{\alpha(\gamma)} = 1, \dots, n_{\alpha(\gamma)}$, $\mu = 1, \dots, n$; hence all exterior derivatives anti-commute, $d\delta_\alpha + \delta_\alpha d = \delta_\alpha \delta_\gamma + \delta_\gamma \delta_\alpha = 0$. Consequently, the m pairs (d, δ_α) which act on the m slices $\Lambda^{n, n_\alpha} \subset \Upsilon$ give rise to m bi-complexes of the form,

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \uparrow \delta_\alpha & & \uparrow \delta_\alpha & & \\ \dots & \xrightarrow{d} & \Lambda^{i, j_\alpha+1} & \xrightarrow{d} & \Lambda^{i+1, j_\alpha+1} & \xrightarrow{d} & \dots \\ & & \uparrow \delta_\alpha & & \uparrow \delta_\alpha & & \\ \dots & \xrightarrow{d} & \Lambda^{i, j_\alpha} & \xrightarrow{d} & \Lambda^{i+1, j_\alpha} & \xrightarrow{d} & \dots \\ & & \uparrow \delta_\alpha & & \uparrow \delta_\alpha & & \\ & & \vdots & & \vdots & & \end{array} \quad (29)$$

where $0 \leq i \leq n$, and $0 \leq j_\alpha \leq n_\alpha$.

Letting all of our bundle objects, in particular the connection 1-forms and the deformation terms, depend also on all group angles, requires a re-formulation of the bundle's covariant exterior derivative: $\mathcal{D} \rightarrow \widehat{\mathcal{D}}$, where

$$\begin{aligned} \widehat{\mathcal{D}}\Psi_T &:= d\Psi_T + \sum_{\alpha=1}^m \left(\delta_\alpha \Psi_T + \omega_\alpha \wedge \Psi_T + (-1)^{\deg(\Psi_T)+1} \Psi_T \wedge \omega_\alpha \right. \\ &\quad \left. + \Omega_\alpha \wedge \Psi_T + (-1)^{\deg(\Psi_T)+1} \Psi_T \wedge \Omega_\alpha \right). \end{aligned} \quad (30)$$

In particular, and after doing some annoying algebra, two successive applications of $\widehat{\mathcal{D}}$ on a generic Ψ_T yields:

$$\widehat{\mathcal{D}}\widehat{\mathcal{D}}\Psi_T = [\mathcal{R}, \Psi]_- + \left[\sum_{\alpha} \mathcal{D}\Omega_\alpha, \Psi_T \right]_- + \left[\sum_{\alpha, \gamma} \delta_\gamma \omega_\alpha, \Psi_T \right]_-$$

$$+ \left[\sum_{\alpha, \gamma} \delta_\gamma \Omega_\alpha, \Psi_T \right]_- + \left[\sum_{\alpha, \gamma} \Omega_\alpha \wedge \Omega_\gamma, \Psi_T \right]_- . \quad (31)$$

(\mathcal{D} in $\mathcal{D}\Omega_\alpha$ is the covariant exterior derivative with respect to the original reduced base).

We shall now associate the deformation elements $\{\Omega_\alpha\}$ with ghost fields (the δ 's turn out to generate ghost numbers - see eq. (36)). This is suggested by the following argument: If we now require that the curvature \mathcal{R}_F remains inert during the extension of the bundle, then the extra four terms in eq. (31) must sum up to zero. Comparing terms of equal “*ghost grading*” we find the following variation laws to hold:

$$\delta_{[\alpha} \Omega_{\gamma]}_+ = -\Omega_{[\alpha} \wedge \Omega_{\gamma]}_+ \quad (32)$$

$$\delta_\alpha \left(\sum_{\gamma=1}^m \omega_\gamma \right) = -\mathcal{D} \Omega_\alpha ; \quad (33)$$

without loss of generality we pick for (32) the variation law $\delta_\alpha \Omega_\gamma = -\Omega_\gamma \wedge \Omega_\alpha$. One now easily verifies that δ_α squares to zero on Ω_γ :

$$\delta_\alpha \delta_\alpha \Omega_\gamma = -(\delta_\alpha \Omega_\gamma) \wedge \Omega_\alpha + \Omega_\gamma \wedge \delta_\alpha \Omega_\alpha = \Omega_\gamma \wedge \Omega_\alpha \wedge \Omega_\alpha - \Omega_\gamma \wedge \Omega_\alpha \wedge \Omega_\alpha = 0, \quad (34)$$

and also on $\omega_F = \sum_\gamma \omega_\gamma$ (recall formula (21)):

$$\begin{aligned} \delta_\alpha \delta_\alpha \omega_F &= -\delta_\alpha (d\Omega_\alpha + \omega_F \wedge \Omega_\alpha + \Omega_\alpha \wedge \omega_F) \\ &= -\mathcal{D} (\Omega_\alpha \wedge \Omega_\alpha) + \mathcal{D} \Omega_\alpha \wedge \Omega_\alpha - \Omega_\alpha \wedge \mathcal{D} \Omega_\alpha = 0. \end{aligned} \quad (35)$$

At this point we already see why it is suggestive to associated the shifts at the gauge sector with ghosts: Eqs. (32)-(33) constitute the BRST algebra associated with the folium F . Note that the sum as a whole, $\omega_F = \sum_\gamma \omega_\gamma$, and not each particular summand, possesses a definite transformation law. Hence, each δ -variation detects a *single-gauge* connection despite the fact that many symmetry structures are involved in our bundle construction.

The coboundary operators $\{\delta_\gamma\}$ can also be interpreted as those operators that generates the deformation elements when applied to the multi-foil basis:

$$\delta_\gamma e_{A_1 \dots A_m} = -\rho_F (\Omega_\gamma)_{A_1 \dots A_m}^{B_1 \dots B_m} e_{B_1 \dots B_m}. \quad (36)$$

On the basis of this definition we may directly re-derive the constraint (32):

$$0 = (\delta_\alpha \delta_\gamma + \delta_\gamma \delta_\alpha) \mathbf{e} = - \left(\delta_{[\alpha} \Omega_{\gamma]}_+ + \Omega_{[\gamma} \wedge \Omega_{\alpha]}_+ \right) \mathbf{e} \Rightarrow \delta_{[\alpha} \Omega_{\gamma]}_+ = -\Omega_{[\gamma} \wedge \Omega_{\alpha]}_+, \quad (37)$$

and the constraint (33):

$$\begin{aligned} 0 &= (\delta_\gamma d + d\delta_\gamma) \mathbf{e} = -\delta_\gamma (\omega_F \mathbf{e}) - d(\Omega_\gamma \mathbf{e}) \\ &= (-\delta_\gamma \omega_F - \mathcal{D}\Omega_\gamma) \mathbf{e} \Rightarrow \delta_\gamma \omega_F = -\mathcal{D}\Omega_\gamma. \end{aligned} \quad (38)$$

In particular, eq. (37) (\equiv (32)) is a generalization of the Maurer-Cartan equation(s) to foliar bundles; the ‘off-diagonal’ equations correspond to cross-fiber interferences.²

A prior knowledge of the extended Maurer-Cartan equations pins down an equivalent (but not self-contained) description for the ghost sector: Let us define the 2-form quantities, $B_{\alpha\gamma} = \delta_\alpha \Omega_\gamma = -\Omega_\gamma \wedge \Omega_\alpha$, $\alpha \neq \gamma$. Now, the nilpotency of δ_α reads: $\delta_\alpha B_{\alpha\gamma} = 0$; on the other hand, $\delta_\gamma B_{\alpha\gamma} = -\llbracket \Omega_\gamma, B_{\alpha\gamma} \rrbracket = -[\Omega_\gamma, B_{\alpha\gamma}]_-$, hence $\delta_\gamma \delta_\gamma B_{\alpha\gamma} = 0$. According to these variation laws, we are dealing here with the B -fields associated with the BRST symmetry on F . However, in contrast with the traditional treating [1, 3], these B -fields are by no means auxiliary degrees of freedom; rather, they are composites made of ghost-ghost pairs.

The variation laws (32)-(33) are manifestly invariant under a duality transformation which is realized by interchanging of labels, $\alpha \leftrightarrow \gamma$, applied simultaneously to both equations. As for the B -fields, the duality manifests itself via transposition. This provides us with an *arbitrary*

²Over the bundle whose basespace is enlarged, the two definitions, (19) and (36), can be combined into

$$\left(d + \sum_{\alpha=1}^m \delta_\alpha \right) \mathbf{e}_{A_1 \dots A_m} = - \sum_{\alpha=1}^m \rho_F \left(\tilde{\omega}_\alpha + \tilde{\Omega}_\alpha \right)_{A_1 \dots A_m}^{B_1 \dots B_m} \mathbf{e}_{B_1 \dots B_m},$$

which we may abbreviate as $(d + \delta_F) \mathbf{e} = -(\omega_F + \Omega_F) \mathbf{e}$. Then, the modified covariant exterior derivative of Ψ_T (formula (30)) takes the succinct form,

$$\hat{\mathcal{D}}\Psi_T = (d + \delta_F) \Psi_T + \llbracket \omega_F + \Omega_F, \Psi_T \rrbracket.$$

classification into ghost-antighost pairs, and with the corresponding pairs of BRST and anti-BRST operators.

Consider for example the 2-folia case $(\alpha, \gamma = 1, 2)$, and put $\delta_1 = \delta$, $\delta_2 = \bar{\delta}$, $\Omega_1 = \Omega$, $\Omega_2 = \Phi$, $\omega_1 = \omega$, and $\omega_2 = \varphi$. Then, from formulas (32) and (33), we have

$$\begin{aligned} \delta\Omega &= -\Omega \wedge \Omega & \delta\Phi &= -\Phi \wedge \Omega & \delta(\omega + \varphi) &= -\mathcal{D}\Omega \\ \bar{\delta}\Phi &= -\Phi \wedge \Phi & \bar{\delta}\Omega &= -\Omega \wedge \Phi & \bar{\delta}(\omega + \varphi) &= -\mathcal{D}\Phi. \end{aligned} \quad (39)$$

In particular, $\delta\Phi + \bar{\delta}\Omega = -\llbracket\Phi, \Omega\rrbracket$. A simultaneous exchange,

$$\delta \leftrightarrow \bar{\delta}, \quad \Omega \leftrightarrow \Phi, \quad (40)$$

transforms the upper triad in (39) into the lower one, and vice versa. Otherwise, we may set $B = -\Phi \wedge \Omega$ and $\bar{B} = -\Omega \wedge \Phi$, whose variation properties are easily read-off from (39),

$$\begin{aligned} \delta B &= 0 & \delta\bar{B} &= -\llbracket\Omega, \bar{B}\rrbracket \\ \bar{\delta}\bar{B} &= 0 & \bar{\delta}B &= -\llbracket\Phi, B\rrbracket \end{aligned} \quad (41)$$

(whence $\delta\bar{\delta}B$ and $\bar{\delta}\delta\bar{B}$ vanish independently). The duality transformation (40) maps a B -field into its dual \bar{B} , and the upper pair in (41) is mapped into the lower one.

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